Name of College - S. S. College J. Bad

Deptt - Mathematics

Topic - Integration of Rational Algebraic Function.

Time - 10.15 A.M To 11.00 A.M

11.00 A.M To 11.45 A.M

Date - 25 - 08 - 2020 Class - B.Sc I (Hons)

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1. Integration by Partial Fraction

$$\int \frac{x^2+1}{x(x^2-1)}\,dx$$

We have $\dfrac{x^2+1}{x(x^2-1)} = \dfrac{x^2+1}{x(x-1)(x+1)}$

Let $\dfrac{x^2+1}{x(x-1)(x+1)} = \dfrac{A}{x} + \dfrac{B}{x-1} + \dfrac{C}{x+1}$

put $x = 0, 1, -1$ $\therefore x^2+1 = A(x^2-1) + Bx(x+1) + Cx(x-1)$

$A = -1$ $B = 1$ $C = 1$

$$\therefore \frac{x^2+1}{x(x-1)(x+1)} = -\frac{1}{x} + \frac{1}{x-1} + \frac{1}{x+1}$$

$$\therefore \int \frac{x^2+1}{x(x-1)(x+1)}\,dx = -\int \frac{dx}{x} + \int \frac{dx}{x-1} + \int \frac{dx}{x+1}$$

$$= -\log x + \log(x-1) + \log(x+1)$$

$$= \log \frac{x^2-1}{x} + C.$$

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2. $\int \frac{x^2+x+2}{(x-2)(x-1)}\,dx$

Here Degree of N^r = Degree of D^r

$\frac{x^2+x+2}{x^2-3x+2} = \frac{(x^2-3x+2)+4x}{x^2-3x+2}$

$x^2-3x+2\,)\,x^2+x+2\,(1$
$\quad x^2-3x+2$
$\quad - \quad + \quad -$
$\quad 4x$

$= 1 + \frac{4x}{(x-1)(x-2)}$

Let $\frac{4x}{(x-1)(x-2)} = \frac{A}{x-1} + \frac{B}{x-2}$

$\Rightarrow 4x = A(x-2) + B(x-1)$

put $x=2$ and $x=1$ respectively

$B=8 \qquad A=-4$

Thus $\frac{x^2+x+2}{(x-2)(x-1)} = 1 - \frac{4}{x-1} + \frac{8}{x-2}$

$\therefore \int \frac{x^2+x+2}{(x-2)(x-1)}\,dx = \int dx - 4\int\frac{dx}{x-1} + 8\int\frac{dx}{x-2}$

$= x - 4\log(x-1) + 8\log(x-2)$

$= x + 8\log(x-2) - 4\log(x-1) + C$

3. $\int \frac{x^3\,dx}{x^2+8x+12}$

Here degree of N^r > Degree of D^r

We have $\frac{x^3}{x^2+8x+12}$

By actual Division

$x^2+8x+12\,)\,\cancel{x^3}\,(x-8$
$\quad \cancel{x^3}+8x^2+12x$
$\quad -8x^2-12x$
$\quad \mp 8x^2 \mp 64x \mp 96$
$\quad 52x+96$

$= x -$

$$\therefore \int \frac{x^3}{x^2+8x+12}\,dx = \int (x-8)\,dx + \int \frac{52x+96}{(x+6)(x+2)}\,dx$$

Now $\dfrac{52x+96}{(x+6)(x+2)} = \dfrac{A}{x+6} + \dfrac{B}{x+2}$

$\Rightarrow 52x+96 = A(x+2) + B(x+6)$

put $x=-2$ and $x=-6$

When $x=-2$

$4B = -104+96 = -8$

$B = -2$

When $x=-6$

$-52\times6+96 = -4A$

$-312+96 = -4A$

$-216 = -4A$

$A = 54$

$$\therefore \int \frac{x^3}{x^2+8x+12}\,dx = \int (x-8)\,dx + 54\int \frac{dx}{x+6} - 2\int \frac{dx}{x+2}$$

$$= \frac{x^2}{2} - 8x + 54\log(x+6) - 2\log(x+2) + c$$

Type 2

When the D^r consists of Repeated and Non-repeated linear factors.

1. $\displaystyle\int \frac{3x+1}{(x-1)^2(x+3)}\,dx$

$$\frac{3x+1}{(x-1)^2(x+3)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+3}$$

$3x+1 = A(x-1)(x+3) + B(x+3) + C(x-1)^2$

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Putting $x = 1$

$4 = 4B \Rightarrow B = 1$

putting $x = -3$

$16C = -8$

$C = -\frac{1}{2}$

Putting $x = 0$

$1 = -3A + 3B + C$

$3A = 3B + C - 1$

$= 3 - \frac{1}{2} - 1$

$= \frac{6 - 1 - 2}{2}$

$= \frac{+3}{2}$

$A = +\frac{1}{2}$

$\therefore \frac{3x+1}{(x-1)^2(x+3)} = +\frac{1}{2(x-1)} + \frac{1}{(x-1)^2} - \frac{1}{2(x+3)}$

$$\therefore \int \frac{3x+1}{(x-1)^2(x+3)} dx = \frac{1}{2}\int \frac{dx}{x-1} + \int \frac{dx}{(x-1)^2} - \frac{1}{2}\int \frac{dx}{x+3}$$

$$= \frac{1}{2}\log(x-1) + \frac{(x-1)^{-2+1}}{-2+1} - \frac{1}{2}\log(x+3)$$

$$= \frac{1}{2}\log(x-1) - \frac{1}{x-1} - \frac{1}{2}\log(x+3) + k$$

Type 3 When D^r contains quadratic factors — as $a_1x^2 + b_1x + c_1$, $a_2x^2 + b_2x + c_2$

We write the given fraction as

$$\frac{Ax+B}{a_1x^2+b_1x+c_1} + \frac{Cx+d}{a_2x^2+b_2x+c_2} + \cdots$$

Integrate $\int \frac{x}{(1+x)(1+x^2)} dx$

Here we have

$$\frac{x}{(1+x)(1+x^2)} = \frac{A}{1+x} + \frac{Bx+C}{1+x^2}$$

$$x = A(1+x^2) + (Bx+C)(1+x)$$

put $x = -1$

$-1 = 2A \Rightarrow A = -1/2$

Equating the co-efficient of x^2

$A + B = 0$

$B = 1/2$

Equating the independent term

$A + C = 0$

$C = -A = 1/2$

Now $$\frac{x}{(1+x)(1+x^2)} = -\frac{1}{2(1+x)} + \frac{\frac{1}{2}x + \frac{1}{2}}{1+x^2}$$

$$= -\frac{1}{2(1+x)} + \frac{1}{2}\frac{x}{1+x^2} + \frac{1}{2}\cdot\frac{1}{1+x^2}$$

$$\int \frac{x}{(1+x)(1+x^2)}dx = -\frac{1}{2}\int\frac{dx}{1+x} + \frac{1}{2\times 2}\int\frac{2x}{1+x^2} + \frac{1}{2}\int\frac{dx}{1+x^2}$$

$$= -\frac{1}{2}\log(1+x) + \frac{1}{4}\log(1+x^2) + \frac{1}{2}\tan^{-1}x + c$$

Into problem $$\int \frac{dx}{1+x+x^2+x^3}$$

$$I = \int \frac{dx}{1+x+x^2+x^3}$$

$$= \int \frac{dx}{(1+x)(1+x^2)}$$

Now

$$\frac{1}{(1+x)(1+x^2)} = \frac{A}{1+x} + \frac{Bx+C}{1+x^2}$$

$$1 = A(1+x^2) + (Bx+C)(1+x)$$

$$= A(1+x^2) + Bx + Bx^2 + C + Cx$$

$$= (A+C) + (A+B)x^2 + (B+C)x$$

Equating co-efficient of x^2

$$A + B = 0 \quad \text{—— (i)}$$

Equating co-efficient of x

$$B + C = 0 \quad \text{—— (ii)}$$

Equating constant

$$A + C = 1 \quad \text{—— (iii)}$$

$$A - C = 0 \quad \Rightarrow 2A = 1$$

$$A + C = 1 \qquad A = \frac{1}{2} \quad \therefore C = A = \frac{1}{2}$$

Now $B + C = 0$

$$B + \frac{1}{2} = 0 \quad B = -\frac{1}{2}$$

$$\therefore \frac{1}{(1+x)(1+x^2)} = \frac{1}{2(1+x)} + \frac{\left(-\frac{1}{2}x + \frac{1}{2}\right)}{1+x^2}$$

$$= \frac{1}{2}\frac{1}{1+x} - \frac{1}{2}\frac{x}{1+x^2} + \frac{1}{2(1+x^2)}$$

$$\therefore \int \frac{dx}{(1+x)(1+x^2)} = \frac{1}{2}\int \frac{dx}{1+x} - \frac{1}{2\times 2}\int \frac{2x}{1+x^2}\,dx + \frac{1}{2}\int \frac{dx}{1+x^2}$$

$$= \frac{1}{2}\log(1+x) - \frac{1}{4}\log(1+x^2) + \frac{1}{2}\tan^{-1}x + C$$

Problem

$$\int \frac{dx}{(x-1)^2(x^2+4)}$$

$$\frac{1}{(x-1)^2(x^2+4)}$$

Let $x-1 = y$

$x = y+1$

$x^2 = (y+1)^2 = y^2+2y+1$

$x^2+4 = y^2+2y+1+4$

Now $$\frac{1}{(x-1)^2(x^2+4)} = \frac{1}{y^2\{(y+1)^2+4\}}$$

$$= \frac{1}{y^2(y^2+2y+5)}$$

$$= \frac{1}{y^2(y^2+2y+5)}$$

~~Now $5+2y+y^2$) 1 ($\frac{1}{5}-\frac{1}{5}$~~

$\cancel{1+\frac{2}{5}y+\frac{1}{5}y^2}$

$-\frac{2}{5}y-\frac{1}{5}y^2$

$5+2y+y^2$) $\cancel{1}$ ($\frac{1}{5}-\frac{2y}{25}$

$\cancel{1+\frac{2}{5}y+\frac{1}{5}y^2}$

$-\frac{2}{5}y-\frac{1}{5}y^2$

$-\frac{2}{5}y - \frac{4y^2}{25} - \frac{2y^2}{25} - \frac{2y^3}{25}$

$+ \quad + \quad + \quad$

$\frac{6y^2}{25} - \frac{1}{5}y^2 \quad -\frac{y^2}{25} + \frac{2y^3}{25}$

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$$= \frac{1}{y^2}\left\{\frac{1}{5} - \frac{2y}{25} - \frac{1}{25}\,\frac{y^2-y^3}{5+2y+y^2}\right\}$$

$$= \frac{1}{5y^2} - \frac{2}{25y} - \frac{1}{25}\,\frac{1-y}{5+2y+y^2}$$

$$\therefore = \frac{1}{5(x-1)^2} - \frac{2}{25(x-1)} - \frac{1}{25}\,\frac{2-x}{x^2+4}$$

$$\therefore \int \frac{dx}{(x-1)^2(x^2+4)} = \frac{1}{5}\int\frac{dx}{(x-1)^2} - \frac{2}{25}\int\frac{dx}{x-1} - \frac{1}{25}\int\frac{2-x}{x^2+4}\,dx$$

$$= -\frac{1}{5(x-1)} - \frac{2}{25}\log(x-1) - \frac{2}{25}\int\frac{dx}{x^2+4} + \frac{1}{25\times2}\int\frac{2x\,dx}{x^2+4}$$

$$= -\frac{1}{5(x-1)} - \frac{2}{25}\log(x-1) - \frac{2}{25}\tan^{-1}\frac{x}{2} + \frac{1}{50}\log(x^2+4) + C$$

Problem

$$\int \frac{dx}{x^3+1}$$

$$\frac{1}{x^3+1} = \frac{1}{(x+1)(x^2-x+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2-x+1}$$

$$\Rightarrow 1 = A(x^2-x+1) + (Bx+C)(x+1)$$

putting $x=-1$

$1 = A(1+1+1) \Rightarrow A = 1/3$

Put $x=0$ $\quad A + C = 1 \quad \therefore C = 1 - 1/3 = 2/3$

Put $x=1$

$1 = A + (B+C)2 \quad \therefore 2B = 1 - A - 2C$

$1 = A + 2B + 2C \quad \therefore B = -1/3 \quad = 1 - 1/3 - \frac{1}{3} - 2/3$

$\therefore \frac{1}{x^3+1} = \frac{1}{3(x+1)} + \frac{-\frac{1}{3}x + 2/3}{x^2-x+1}$

$$\int \frac{1}{x^3+1}\,dx = \frac{1}{3}\int \frac{dx}{x+1} - \frac{1}{3}\int \frac{x-2}{x^2-x+1}\,dx$$

$$= \frac{1}{3}\int \frac{dx}{x+1} - \frac{1}{6}\int \frac{2x-1-3}{x^2-x+1}\,dx$$

$$= \frac{1}{3}\int \frac{dx}{x+1} - \frac{1}{6}\int \frac{2x-1}{x^2-x+1}\,dx + \frac{1}{2}\int \frac{dx}{x^2-x+1}$$

$$= \frac{1}{3}\log(x+1) - \frac{1}{6}\log(x^2-x+1) + \frac{1}{2}\int \frac{dx}{(x-\frac{1}{2})^2+\frac{3}{4}}$$

$$= \frac{1}{3}\log(x+1) - \frac{1}{6}\log(x^2-x+1) + \frac{1}{2}\cdot\frac{1}{\sqrt{3}/2}\tan^{-1}\frac{x-\frac{1}{2}}{\sqrt{3}/2}$$

$$= \frac{1}{3}\log(x+1) - \frac{1}{6}\log(x^2-x+1) + \frac{1}{\sqrt{3}}\tan^{-1}\frac{2x-1}{\sqrt{3}} + C$$

Do your self $\int \frac{dx}{x^3-1}$